Note on Chern-Simons term correction to holographic entanglement entropy

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ABSTRACT: From the perspective of AdS/CFT correspondence, we study the gravitational Chern-Simons term correction to the holographic entanglement entropy of CFT on the conformal boundary of asymptotically AdS$_3$ spacetime using the off-shell Euclidean path integral method. We show that, like the BTZ black hole entropy, the holographic entanglement entropy is indeed modified due to the gravitational Chern-Simons term although the bulk geometry does not change.

Contents

1 Introduction

The entanglement entropy (often defined as the von Neumann entropy), which is used to describe the correlations between two (or more) subsystems of a quantum field system, has received a lot of attention over the past few years. It plays an important role in studying the critical phenomena or phase transitions in condensed matter physics and quantum field theory (for a recent review, see, e.g., [1]). Since it also describes the lack of information for an observer in a subsystem A who is inaccessible to other subsystems, the concept of entanglement entropy is also applied to black hole physics, where the black hole entropy is viewed as an entanglement entropy resulting from the entanglement of the vacuum states of certain quantum fields inside and outside the event horizon of black hole [2–5].

From the prospective of holography [6–10], the entanglement entropy acquires new understanding. In ref. [11], Ryu and Takayanagi proposed the concept of holographic entanglement entropy (HEE): the entanglement entropy of boundary CFT can be calculated by considering bulk geometry of AdS space. They considered CFT on an d-dimensional flat spacetime which serves as the (conformal) boundary of an $(d + 1)$-dimensional bulk anti-de Sitter spacetime (AdS_{d+1}) and proposed that the entanglement entropy S_A of an d-dimensional CFT system in a region A in the flat spacetime is determined by an $(d-1)$-dimensional spacelike, static, minimal hypersurface γ_A in the bulk with ∂A as its boundary via

$$S_A = \frac{\mathcal{A}}{4G_{d+1}}, \tag{1.1}$$

where the conformal field in the region A is a subsystem of the whole system in $A + B$ in the flat spacetime, $\mathcal{A}$ is the area of γ_A, and G_{d+1} is the $(d + 1)$-dimensional Newton

gravitational constant. Note that eq. (1.1) has the same form as the black hole area entropy formula. This is the reason why they call S_A the holographic entanglement entropy. In ref. [11], they checked the validity of this relation by a few examples. For the case of AdS$_3$/CFT$_2$ correspondence, the entanglement entropy calculated from the well-known 2-dimensional CFT matches eq. (1.1) very well. When it is applied to higher dimensions, e.g., AdS$_5$/CFT$_4$, however, the entropies calculated from both sides are not exactly the same. It is argued that the occurrence of this discrepancy is due to the dual description of the classical gravity in bulk AdS is a strongly coupled supersymmetric Yang-Mills theory, but not a free one, while they only considered the free field approximation in the side of field theory. This is similar to the case of the well-known difference by a factor 3/4 between the entropy of black D3-branes and the entropy of free supersymmetric Yang-Mills theory.

Shortly, the proposal has been extended to many cases such as calculating the entropy of black holes [12], de Sitter brane world [13] and studying the confinement/deconfinement transition of gauge fields [14], and others, see for example ref. [15]. In the mean time, the proposal has been studied in more detail in [16, 17] and Fursaev [18] has given a proof for the proposal by applying the Euclidean path integral approach [19] to Riemann surfaces with conical singularities [20] in the AdS/CFT correspondence.

Note that the proposal relates the bulk geometry to the HEE of boundary CFT and in the AdS/CFT corresponence, the higher order curvature terms in the bulk are dual to the large N expansion corrections in the dual CFT, it is natural to consider higher order corrections to the bulk gravitational action and to see how the HEE of boundary CFT gets modified. Typical higher order corrections to the Hilbert-Einstein action are Gauss-Bonnet term [21] and Chern-Simons term [22] in the low energy effective action of string theory. The Gauss-Bonnet term correction to HEE has been briefly discussed in [18]. Now an interesting question is to see how the Chern-Simons term correction modifies the HEE of dual CFT. It is interesting because in general, the presence of the gravitational Chern-Simons term modifies the bulk equation of motion by the Cotton tensor. However, for the global AdS$_3$ spacetime and the BTZ black hole, the Cotton tensor vanishes identically, i.e., the bulk geometry does not change and a direct use of eq. (1.1) leads no modification. It does not mean that the Chern-Simons term correction is trivial in this case, because it violates the diffeomorphism invariance of the bulk gravitational action. In the AdS/CFT correspondence, the violation of diffeomorphism invariance in asymptotically AdS$_3$ (AAdS$_3$) spacetime causes gravitational anomaly (covariant anomaly) in the dual 2-dimensional CFT on the conformal boundary and changes the central charges of the CFT, this corresponds to a modification to the BTZ black hole entropy by a term proportional to its inner horizon [25–27]. The gravitational Chern-Simons term correction to black hole entropy has also been studied in [28, 29] using the Wald's entropy formula [30]. Although in this case the variation of the Chern-Simons term doesn't modify the bulk geometry, it does modify the black hole entropy. The relation between trace anomalies and HEE has been studied in [31].

The purpose of this paper is to study how the gravitational Chern-Simons term modifies the HEE in AAdS$_3$ spacetime. The standard way to calculate its entanglement entropy is the replica trick [1] which computes the partition function of the quantum fields in

Euclidean path integral formulation on an n-sheeted Riemann surface, in the limit of $n \to 1$. The conical singularity is introduced as a result of a conformal coordinate transformation on the n-sheeted Riemann surface, which produces a δ function in the Riemann tensor. Then using the off-shell Euclidean path integral method [18], we find that to the leading order the gravitational Chern-Simons term indeed varies the HEE (when the bulk is BTZ black hole) as expected. We point out here that this is a approximately calculation since we do not actually solve the constraint condition for γ_A.

The paper is organized as follows. In section II, we give a brief introduction of the replica trick and the off-shell method introduced in [1, 18]. In section III we calculate the gravitational Chern-Simons term correction to the HEE in the cases of zero temperature and finite temperature, respectively. Then in section IV we make some conclusions and discussions.

2 The replica trick and the off-shell method

2.1 Replica trick

Let's consider an d-dimensional CFT $\phi(x)$ defined on a flat spactime $\mathbb{R}^{1,d-1}$. Divide the CFT system into two subsystems A and B with boundary ∂_A. The density matrix ρ of the whole system $A + B$ in a thermal state at inverse temperature β can be expressed in the Euclidean path integral as [1]

$$\rho_{\phi''\phi'} = \frac{1}{Z[\beta]} \int \mathcal{D}\phi(x) \prod_x \delta(\phi(\vec{x},0) - \phi'(\vec{x},0)) \prod_x \delta(\phi(\vec{x},\beta) - \phi''(\vec{x},0)) e^{-I_E}, \quad (2.1)$$

where I_E is the Euclidean action of the whole system and the unitarity requires $tr\rho = 1$. Let the subsystem A consists of points x in some region. The reduced density matrix $\rho_A (= \mathrm{tr}_B \rho)$ for the subsystem A is obtained by sewing together the points which only belong to B. Then the entanglement entropy of A is calculated from the von Neumann entropy

$$S_A = -\mathrm{tr}_A \rho_A \ln \rho_A, \quad (2.2)$$

To compute S_A, define a new quantity $\mathrm{tr}\rho_A^n \equiv Z_n(A)/Z^n$, $Z_n(A)$ is the partition function over an n-sheet structure by making n copies of the fields and sewing them together through $\phi'(x)_k = \phi''(x)_{k+1}$ and $\phi'(x)_1 = \phi''(x)_n$ $(1 \le k \le n)$.[1] Then S_A is given by

$$S_A = -\lim_{n \to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}. \quad (2.3)$$

Note that in the limit of $n \to 1$, eq. (2.3) can be expressed as the Tsallis entropy [32]

$$S_A = \lim_{n \to 1} \frac{\mathrm{tr}\rho_A^n - 1}{1 - n}. \quad (2.4)$$

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[1]As has been mentioned in [31], there is another sewing condition which needs $\phi(x)$ to be continuous at boundary ∂_A.

2.2 Off-shell method

Following Subsection A, the d-dimensional CFT is defined on

$$ds^2 = d\tau^2 + dx^2 + h_{ij}dy^i dy^j, \tag{2.5}$$

where τ is the Euclidean time, h_{ij} is the spatial component of the metric and $i, j \in \{2,3,\cdots, d-1\}$. Recall that in 2-dimensional space, the conformal transformations can be realized as the holomorphic coordinate transformations in complex plane [33]. So we can define

$$w = \tau + ix \quad \text{and} \quad \bar{w} = \tau - ix. \tag{2.6}$$

The n-sheet structure mentioned above is realized by a singular coordinate transformation [31, 34]

$$w \to z = w^{\frac{1}{n}}, \tag{2.7}$$

it is singular at $z = 0$ (when $n \neq 1$), which is the boundary between A and B at $\tau = 0$. Then eq. (2.5) becomes

$$ds^2 = n^2 z^{n-1} \bar{z}^{n-1} dz d\bar{z} + h_{ij}dy^i dy^j. \tag{2.8}$$

Let $n = 1 + \epsilon$, ϵ is an infinitesimal real parameter. To the linear order of ϵ, eq. (2.8) is

$$\begin{aligned} ds^2 &= (1 + 2\epsilon + \epsilon \ln(z\bar{z}))dz d\bar{z} + h_{ij}dy^i dy^j + \mathcal{O}(\epsilon^2) \\ &\equiv 2h_{z\bar{z}}dz d\bar{z} + h_{ij}dy^i dy^j + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.9}$$

A direct calculation shows that the Riemann tensor contains singular components due to the $\ln(z\bar{z})$ term in eq. (2.9)

$$\Gamma_{\bar{z}\bar{z}}^{\bar{z}} = h^{z\bar{z}} \frac{\partial h_{z\bar{z}}}{\partial \bar{z}} \quad \text{and} \quad \Gamma_{zz}^{z} = h^{z\bar{z}} \frac{\partial h_{z\bar{z}}}{\partial z}, \tag{2.10}$$

$$R_{zz\bar{z}}^{z} = \frac{\partial \Gamma_{z\bar{z}}^{z}}{\partial z} - \frac{\partial \Gamma_{zz}^{z}}{\partial \bar{z}} + \Gamma_{zz}^{z}\Gamma_{z\bar{z}}^{z} - \Gamma_{z\bar{z}}^{z}\Gamma_{zz}^{z} = -\frac{\partial \Gamma_{zz}^{z}}{\partial \bar{z}},$$

$$R_{\bar{z}\bar{z}z}^{\bar{z}} = \frac{\partial \Gamma_{\bar{z}z}^{\bar{z}}}{\partial \bar{z}} - \frac{\partial \Gamma_{\bar{z}\bar{z}}^{\bar{z}}}{\partial z} + \Gamma_{\bar{z}\bar{z}}^{\bar{z}}\Gamma_{z\bar{z}}^{\bar{z}} - \Gamma_{z\bar{z}}^{\bar{z}}\Gamma_{\bar{z}\bar{z}}^{\bar{z}} = -\frac{\partial \Gamma_{\bar{z}\bar{z}}^{\bar{z}}}{\partial z}. \tag{2.11}$$

The singular part of Riemann tensor is

$$^sR_{zz\bar{z}}^{z} = {}^sR_{\bar{z}\bar{z}z}^{\bar{z}} = -2\pi\epsilon\delta^{(2)}(z, \bar{z}), \tag{2.12}$$

where the following relations have been used

$$\begin{aligned} R_{zz\bar{z}}^{z} = R_{z\bar{z}} &= -h^{z\bar{z}} \frac{\partial^2 h_{z\bar{z}}}{\partial z \partial \bar{z}} - \frac{\partial h^{z\bar{z}}}{\partial \bar{z}} \frac{\partial h_{z\bar{z}}}{\partial z} \\ &= -2(1 - 2\epsilon - 2\epsilon \ln(z\bar{z}))\frac{\epsilon}{2}2\pi\delta^{(2)}(z, \bar{z}) - \frac{\partial h^{z\bar{z}}}{\partial \bar{z}} \frac{\partial h_{z\bar{z}}}{\partial z} \\ \text{and} \quad \frac{\partial^2 \ln(z\bar{z})}{\partial z \partial \bar{z}} &= 2\pi\delta^{(2)}(z, \bar{z}). \end{aligned} \tag{2.13}$$

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Then the singular part of Ricci scalar is

$$^sR = 2h^{z\bar{z}}R_{z\bar{z}} = -4\pi\epsilon h^{z\bar{z}}\delta^{(2)}(z,\bar{z}). \tag{2.14}$$

Now the key point is that in the AdS/CFT correspondence [8–10], the partition functions of the bulk (super)gravity $Z_{gr}(h)$ and boundary CFT $Z_{\text{CFT}}(h)$ can be related through

$$Z_{gr}(h) = Z_{\text{CFT}}(h), \tag{2.15}$$

and $Z_{gr}(h)$ can be calculated from its Euclidean path integral as $Z_{gr}(h) = \int \mathcal{D}[g]\exp(-I_E(g))$, where g is the bulk metric with induced metric h on the boundary. For 3-dimensional Einstein gravity with a negative cosmological constant, the Einstein-Hilbert action with Gibbons-Hawking surface term is

$$I_{gr} = \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}d^3x\left(R + \frac{2}{l_a^2}\right) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{|h|}d^2xK, \tag{2.16}$$

where l_a is the radius of AdS$_3$ spacetime and K is the extrinsic curvature on the boundary $\partial\mathcal{M}$. The dual field theory on $\partial\mathcal{M}$ is a 2-dimensional CFT on a circle consisting of subsystems A and B. Thanks to the n-sheeted structure on the boundary, it introduces a discrete parameter $\beta = 2\pi n$ in the partition function which reads [18]

$$Z_{\text{CFT}}(h,T) = \lim_{\beta\to 2\pi} Z_{\text{CFT}}(\beta,h,T), \tag{2.17}$$

where T is the temperature of the CFT system. Analogous to the treatment in canonical ensemble, the entanglement entropy of A is given by [37]

$$S_A = \lim_{\beta\to 2\pi}\left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z_{\text{CFT}}(\beta,h,T). \tag{2.18}$$

With eq. (2.15), S_A can be calculated by

$$\begin{aligned}
S_A &= \lim_{\beta\to 2\pi}\left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z_{\text{CFT}}(h)\\
&= \lim_{\beta\to 2\pi}\left(1 - \beta\frac{\partial}{\partial\beta}\right)\left(-I_E^{(0)}(g_0) - I_E^{(2)}(g_0) + \cdots\right),
\end{aligned} \tag{2.19}$$

where $I_E^{(0)}(g_0)$ indicates the (dominant) contribution from the solution g_0 (of bulk dynamical equation), and $I_E^{(1)}(g_0) = 0$, i.e., bulk dynamical equation has been used. In the saddle point approximation (zero loop approximation), eq. (2.19) can be well approximately computed as

$$S_A = \lim_{\beta\to 2\pi}\left(1 - \beta\frac{\partial}{\partial\beta}\right)(-I_E^{(0)}(g_0)). \tag{2.20}$$

Dividing the the gravitational action (2.16) into the regular part and singular part and using eq. (2.14) we obtain

$$\begin{aligned}
iI_{gr} = -I_E &= \frac{i}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}d^3x\left(^rR + \frac{2}{l_a^2}\right) + \frac{i}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{|h|}d^2xK\\
&\quad + \frac{1}{16\pi G}\int_{\mathcal{M}}h_{z\bar{z}}\sqrt{\gamma}dzd\bar{z}dx^sR\\
&= \frac{i}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}d^3x\left(^rR + \frac{2}{l_a^2}\right) + \frac{i}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{|h|}d^2xK - \frac{\epsilon}{4G}\int_{\Sigma}\sqrt{\gamma}dx,
\end{aligned} \tag{2.21}$$

where γ is the induced metric of the spatial section Σ with codimension-2. The term $\int_\Sigma \sqrt{\gamma} dx$ is the Dirac-Nambu-Goto like action which denotes the area $\mathcal{A}$ of the codimension-2 surface. Then we obtain

$$S_A = \frac{1}{4G} \int_\Sigma \sqrt{\gamma} dx = \frac{\mathcal{A}}{4G}. \tag{2.22}$$

The constraint equation $\delta \mathcal{A} = 0$ requires $\mathcal{A}$ be the bulk minimal surface with the boundary ∂_A. Eq. (2.22) is just of the form of eq. (1.1) in AdS$_3$ spacetime. The above discussion can be generalized into higher dimensional spacetimes.

3 Gravitational Chern-Simons term corrections to HEE

The action of 3-dimensional Einstein gravity with a negative cosmological constant in the presence of a gravitational Chern-Simons term is

$$
\begin{aligned}
I &= I_{\text{EH}} + I_{\text{CS}} \\
&= \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} d^3 x \left(R + \frac{2}{l_a^2} \right) + \frac{1}{8\pi G} \int_{\partial \mathcal{M}} \sqrt{|h|} d^2 x K \\
&\quad + \frac{\alpha}{32\pi G} \int_{\mathcal{M}} \sqrt{-g} d^3 x \left(\Gamma^\alpha_{\beta\mu} \frac{\partial \Gamma^\beta_{\alpha\lambda}}{\partial x^\nu} + \frac{2}{3} \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\gamma\nu} \Gamma^\gamma_{\alpha\lambda} \right) \epsilon^{\mu\nu\lambda},
\end{aligned} \tag{3.1}
$$

where α is the coupling constant, h is the induced metric on the boundary, and K is the extrinsic curvature of the boundary. Gravity described by this action is called the topologically massive gravity (TMG) [22] and it recently received much attention [39–49]. Variation of I yields equation of motion

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - \frac{1}{l_a^2} g_{\alpha\beta} + \alpha C_{\alpha\beta} = 0, \tag{3.2}$$

where

$$C_{\alpha\beta} = \epsilon_\alpha{}^{\mu\nu} \nabla_\mu \left(R_{\nu\beta} - \frac{1}{4} g_{\nu\beta} R \right) \tag{3.3}$$

is called Cotton tensor.

TMG permits the pure AdS$_3$ spacetime and BTZ black hole as its solutions and the Cotton tensor vanishes identically in these cases. Here we only focus on these cases.

For the AdS$_3$ spacetime and BTZ black hole, a direct use of eq. (1.1) reflects no correction to HEE in the presence of the gravitational Chern-Simons term. Since the bulk geometries do not change in those cases. However, it does not mean that the Chern-Simons term correction is trivial in these examples. Because the presence of gravitational Chern-Simons term violates the diffeomorphism invariance of the gravitational action. In the AdS/CFT correspondence, the gravitational anomaly (covariant anomaly) occurs in the dual 2-dimensional CFT and modifies its central charges as [25]

$$c_L = c \left(1 + \frac{\alpha}{l_a} \right) \quad \text{and} \quad c_R = c \left(1 - \frac{\alpha}{l_a} \right), \tag{3.4}$$

where c_L and c_R are the left-moving and right-moving central charges, respectively, and $c = 3l_a/(2G)$. As a result, it can be seen that the gravitational anomaly is reflected by a modification to the BTZ black hole entropy by a term proportional to its inner horizon [26]

$$S_{\mathrm{BTZ}} = \frac{\pi r_+}{2G} - \frac{\pi \alpha r_-}{2G l_a} = \frac{\pi c}{3 l_a} \left(r_+ - \frac{\alpha r_-}{l_a} \right), \tag{3.5}$$

where r_+ and r_- are outer and inner horizons of the BTZ black hole, respectively. The same result are obtained using other methods [28, 29].

Note that in the AdS/CFT correspondence, the BTZ black hole entropy is identical to the entropy of its dual 2-dimensional CFT on the conformal boundary at spatial infinity. Thus it is natural to consider how the gravitational Chern-Simons term modifies the HEE of dual CFT. To see this, recall that the solutions of Einstein equation which are AAdS spacetime can be expanded through the Fefferman-Graham (FG) expansion in the Gaussian normal coordinates [23–25, 38]

$$ds^2 = d\rho^2 + l_a^2 h_{ij} dx^i dx^j, \tag{3.6}$$

where

$$h_{ij} = e^{2\rho/l_a} h_{ij}^{(0)} + h_{ij}^{(2)} + e^{-2\rho/l_a} h_{ij}^{(4)} + \cdots, \tag{3.7}$$

The conformal boundary of AAdS spacetime is located at $\rho \to \infty$, and the induced metric on it is $h_{ij}^{(0)}$. The stress energy tensor of the dual 2-dimensional CFT is [25, 35, 36]

$$T_{ij} = \frac{1}{8\pi G l_a} (h_{ij}^{(2)} - h_{(0)}^{kl} h_{kl}^{(2)} h_{ij}^{(0)}). \tag{3.8}$$

It can be seen that although the CFT is defined on the boundary with metric $h_{ij}^{(0)}$, its stress energy tensor relates to the bulk metric up to $h_{ij}^{(2)}$. Besides, the contribution from the gravitational Chern-Simons term to T_{ij} also include $h_{ij}^{(2)}$. This indicates that to study the gravitational Chern-Simons term correction to HEE, we need to take the contribution of $h_{ij}^{(2)}$ into consideration. It is shown that like the HEE, the correction can also be expressed as the geometric quantities like the metric and Christoffel symbol.[2]

3.1 Zero temperature CFT case

The zero temperature 2-dimensional CFT case corresponds to the bulk global AdS$_3$ spacetime with metric

$$\begin{aligned} ds^2 &= d\rho^2 + l_a^2 \left(-\cosh^2 \frac{\rho}{l_a} dt^2 + \sinh^2 \frac{\rho}{l_a} d\phi^2 \right) \\ &= d\rho^2 + e^{2\rho/l_a} \left(-\frac{l_a^2}{4} dt^2 + \frac{l_a^2}{4} d\phi^2 \right) - \frac{l_a^2}{2} dt^2 - \frac{l_a^2}{2} d\phi^2 + e^{-2\rho/l_a} \left(-\frac{l_a^2}{4} dt^2 + \frac{l_a^2}{4} d\phi^2 \right). \end{aligned} \tag{3.9}$$

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[2]See appendix for the detailed calculation.

Its dual 2-dimensional CFT is defined on the boundary at large $\rho = \rho_0 \gg l_a$ with conformal metric $h_{ij}^{(0)}$, ρ_0 is related to the UV cutoff a of the CFT. To use the replica trick in section II, we work in the Euclidean signature by taking $t \to -i\tau$. Then define

$$w = \frac{l_a}{2}\tau + i\frac{l_a}{2}\phi = z^n = z^{1+\epsilon}, \tag{3.10}$$

As has been mentioned above that we need to take into account of the $h_{ij}^{(2)}$ terms. So in the $\rho = \rho_0 \gg l_a$ limit eq. (3.9) can be approximately described by

$$\begin{aligned}
ds^2 &\simeq d\rho^2 + e^{2\rho/l_a}(1 + 2\epsilon + \epsilon \ln(z\bar{z}))dzd\bar{z} \\
&\quad + (1 + 2\epsilon + 2\epsilon \ln z)dz^2 + (1 + 2\epsilon + 2\epsilon \ln \bar{z})d\bar{z}^2 + \mathcal{O}(\epsilon^2) \\
&= d\rho^2 + 2e^{2\rho/l_a}h_{z\bar{z}}dzd\bar{z} + h_{zz}dz^2 + h_{\bar{z}\bar{z}}d\bar{z}^2 + \mathcal{O}(\epsilon^2) \\
&\equiv d\rho^2 + 2g_{z\bar{z}}dzd\bar{z} + h_{zz}dz^2 + h_{\bar{z}\bar{z}}d\bar{z}^2 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.11}$$

Thus we obtain

$$\begin{aligned}
g &= h_{zz}h_{\bar{z}\bar{z}} - e^{4\rho/l_a}h_{z\bar{z}}h_{\bar{z}z} \simeq -\frac{e^{4\rho/l_a}}{4}(1 + 4\epsilon + 2\epsilon \ln(z\bar{z})) + \mathcal{O}(\epsilon^2), \\
\Gamma_{\bar{z}\bar{z}}^{\bar{z}} &= 4(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{z\bar{z}}\frac{\partial h_{z\bar{z}}}{\partial \bar{z}} + \frac{1}{2}h^{\bar{z}z}\frac{\partial h_{\bar{z}\bar{z}}}{\partial \bar{z}} + \mathcal{O}(\epsilon^2), \\
\Gamma_{zz}^{z} &= 4(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{z\bar{z}}\frac{\partial h_{z\bar{z}}}{\partial z} + \frac{1}{2}h^{zz}\frac{\partial h_{zz}}{\partial z} + \mathcal{O}(\epsilon^2), \\
\Gamma_{\rho\bar{z}}^{\bar{z}} &= \Gamma_{\rho z}^{z} = \frac{1}{l_a}(1 - 4\epsilon - 2\epsilon \ln(z\bar{z})) + \mathcal{O}(\epsilon^2), \\
\Gamma_{z\bar{z}}^{\rho} &= \Gamma_{\bar{z}z}^{\rho} = -e^{2\rho/l_a}\frac{h_{z\bar{z}}}{l_a} \quad \text{and} \quad \Gamma_{\bar{z}\rho}^{z} = -\frac{4}{l_a}e^{-2\rho/l_a}(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{\bar{z}\bar{z}}h_{z\bar{z}} + \mathcal{O}(\epsilon^2), \\
\Gamma_{\rho z}^{\bar{z}} &= -\frac{4}{l_a}e^{-2\rho/l_a}(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{zz}h_{\bar{z}z} + \mathcal{O}(\epsilon^2), \\
\Gamma_{zz}^{\bar{z}} &= -4e^{-2\rho/l_a}(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{zz}\frac{\partial h_{z\bar{z}}}{\partial z} + \frac{1}{2}g^{\bar{z}z}\frac{\partial h_{zz}}{\partial z} + \mathcal{O}(\epsilon^2), \\
\Gamma_{\bar{z}\bar{z}}^{z} &= -4e^{-2\rho/l_a}(1 - 4\epsilon - 2\epsilon \ln(z\bar{z}))h_{\bar{z}\bar{z}}\frac{\partial h_{z\bar{z}}}{\partial \bar{z}} + \frac{1}{2}g^{z\bar{z}}\frac{\partial h_{\bar{z}\bar{z}}}{\partial \bar{z}} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.12}$$

From eq. (2.13), the singular part of the Riemann tensor and Riemann scalar are

$$\begin{aligned}
{}^sR_{zz\bar{z}}^{z} &= {}^sR_{\bar{z}\bar{z}z}^{\bar{z}} = -2\pi\epsilon\delta^{(2)}(z,\bar{z}) \quad \text{and} \quad {}^sR_{zz\bar{z}}^{\bar{z}} = {}^sR_{\bar{z}\bar{z}z}^{z} = 4\pi\epsilon e^{-2\rho/l_a}\delta^{(2)}(z,\bar{z}), \\
{}^sR &= -8\pi\epsilon e^{-2\rho/l_a}\delta^{(2)}(z,\bar{z}) + 32\pi\epsilon e^{-6\rho/l_a}\delta^{(2)}(z,\bar{z}) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.13}$$

Like eq. (2.21), the Einstein-Hilbert action is divided into two parts

$$I_{\text{EH}} = {}^rI_{\text{EH}} + \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}dzd\bar{z}dx^s R, \tag{3.14}$$

and the contribution to the HEE is

$$\begin{aligned}
S_{\text{EH}} &= \frac{1}{4G}\int_0^{\rho_0}d\rho - \frac{1}{G}\int_0^{\rho_0}e^{-4\rho/l_a}d\rho \\
&= \frac{c}{6}\ln\frac{L}{a} - \frac{c}{6} + \mathcal{O}\left(\left(\frac{a}{L}\right)^4\right) \simeq \frac{c}{6}\ln\frac{L}{a} - \frac{c}{6}.
\end{aligned} \tag{3.15}$$

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The first term is just the universal term of the HEE obtained in [1, 11, 37].[3] Here ρ_0 is related to the UV cutoff a of the dual 2-dimensional CFT by $\exp(\rho_0/l_a) = L/a$, and L is the length of the whole system $A + B$.

To compute the correction of the gravitational Chern-Simons term to HEE, we can use the result in [20, 26], the introduction of the singular coordinate transformation in eq. (2.7) causes conical singularities on the bulk minimal surface. As a result, the induced Riemann tensor on the minimal surface is

$$R^{\mu\nu}{}_{\alpha\beta} = {}^r R^{\mu\nu}{}_{\alpha\beta} - 2\pi\epsilon[(n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha)]\delta_\Sigma. \tag{3.16}$$

where δ_Σ is a 2-d delta function which is nonzero on the bulk minimal surface and $n^\mu n_\alpha = n_1^\mu n_\alpha^1 + n_2^\mu n_\alpha^2$. Then the Chern-Simons term is also divided into two parts

$$I_{\rm CS} = {}^r I_{\rm CS} - \frac{\alpha\epsilon}{32G} \int_{\mathcal{M}} d^3x\sqrt{-g}\Gamma_{\alpha\beta\mu}\epsilon^{\mu\nu\lambda}[(n^\beta n_\nu)(n^\alpha n_\lambda) - (n^\beta n_\lambda)(n^\alpha n_\nu)]\delta_\Sigma, \tag{3.17}$$

where ${}^r I_{\rm CS}$ is the regular part of the gravitational Chern-Simons term. From the appendix, the singular part of eq. (3.17) is

$$^s I_{\rm CS} = \frac{\alpha\epsilon}{32G} \int_\Sigma dx\sqrt{\gamma}\frac{\epsilon^{z\bar{z}\rho}}{2}(h_{tt} - h_{xx}),_\rho (\hat{n}_1^\rho)^2 \left[\frac{h_{tx}}{h}(h_{tt}+h_{xx}) + \frac{h_{tx}}{h}\left(h_{xx} + \frac{h_{tx}^2}{h_{xx}} \right) \right] \frac{1}{|\frac{dF}{d\rho}|}. \tag{3.18}$$

Since h_{tx} is zero in eq. (3.9), $^s I_{\rm CS}$ vanishes, consequently, $S_{\rm CS} = 0$. Thus we can see that the gravitational Chern-Simons term does not correct the HEE of the 2-d CFT (at zero temperature) on the boundary of global AdS$_3$ spacetime.

3.2 Finite temperature CFT case

The finite temperature 2-dimensional CFT case corresponds to the case of the bulk having a BTZ black hole with metric [50]

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l_a^2 r^2}dt^2 + \frac{l_a^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 + r^2\left(d\phi - \frac{r_+ r_-}{l_a r^2}dt \right)^2. \tag{3.19}$$

The black hole is of mass $m = (r_+^2 + r_-^2)/(8Gl_a^2)$, angular momentum $j = r_+ r_-/(4Gl_a)$ and temperature $T = (r_+^2 - r_-^2)/(2\pi r_+ l_a^2)$. It can also be expressed in the Gaussian normal coordinates by setting

$$d\rho = \frac{l_a r dr}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}, \tag{3.20}$$

which gives

$$l_a e^{\rho/l_a} = \sqrt{r^2 - r_+^2} + \sqrt{r^2 - r_-^2}. \tag{3.21}$$

---

[3]Compared to the calculation in field theory side, there is a difference by a factor 2. This is due to the fact that here we only considered one brunch point, while there are two brunch points at the boundary between A and B. With this eq. (3.15) can reproduce the correct result.

In the $\rho \gg l_a$ limit, we have

$$e^{2\rho/l_a} = \frac{4r^2}{l_a^2} - 16Gm - \frac{16G^2l_a^2m^2}{r^2} + \frac{16G^2j}{r^2} + \mathcal{O}(r^{-4})$$

$$\text{and} \quad r^2 = \frac{l_a^2}{4}e^{2\rho/l_a} + 4l_a^2Gm + 4e^{-2\rho/l_a}(4G^2l_a^2m^2 - G^2j^2). \tag{3.22}$$

Then the metric can be expanded as

$$\begin{aligned}
ds^2 &= d\rho^2 + \left(-\frac{1}{4}e^{2\rho/l_a} + 4Gm\right)dt^2 \\
&\quad + \left(\frac{l_a^2}{4}e^{2\rho/l_a} + 4Gl_a^2m + 4e^{-2\rho/l_a}(4G^2l_a^2m^2 - G^2j^2)\right)d\phi^2 \\
&\quad - 8Gjd\phi dt - 4e^{-2\rho/l_a}\left(4G^2m^2 - \frac{G^2j^2}{l_a^2}\right)dt^2 + \mathcal{O}(e^{-4\rho/l_a}) \\
&= d\rho^2 + e^{2\rho/l_a}\left(-\frac{1}{4}dt^2 + \frac{l_a^2}{4}d\phi^2\right) + 4Gmdt^2 + 4Gl_a^2md\phi^2 - 8Gjd\phi dt \\
&\quad + e^{-2\rho/l_a}\left(\left(4\frac{G^2j^2}{l_a^2} - 16G^2m^2\right)dt^2 + 16G^2l_a^2m^2d\phi^2 - 4G^2j^2d\phi^2\right) + \mathcal{O}(e^{-4\rho/l_a}).
\end{aligned} \tag{3.23}$$

In this case, we need to take $t \to -i\tau$, $j \to ij$ and define

$$w = \frac{1}{2}\tau + i\frac{l_a}{2}\phi = z^n = z^{1+\epsilon}. \tag{3.24}$$

After reserving the $h_{ij}^{(2)}$ terms we obtain

$$ds^2 = d\rho^2 + 2e^{2\rho/l_a}h_{z\bar{z}}dzd\bar{z} + \left(i\frac{8Gj}{l_a} - 8Gm\right)h_{zz}dz^2 - \left(i\frac{8Gj}{l_a} + 8Gm\right)h_{\bar{z}\bar{z}}d\bar{z}^2 + \mathcal{O}(\epsilon^2), \tag{3.25}$$

where $h_{z\bar{z}}$, h_{zz} and $h_{\bar{z}\bar{z}}$ are the same as those in the pure AdS$_3$ case. Thus just following the procedure above, the singular part of the Riemann tensors and Ricci scalar are

$$\begin{aligned}
{}^sR^z_{zz\bar{z}} &= {}^sR^{\bar{z}}_{\bar{z}\bar{z}z} = -2\pi\epsilon\delta^{(2)}(z,\bar{z}), \\
{}^sR^{\bar{z}}_{zz\bar{z}} &= 4\pi\epsilon e^{-2\rho/l_a}\left(i\frac{8Gj}{l_a} - 8Gm\right)\delta^{(2)}(z,\bar{z}) \\
\text{and} \quad {}^sR^z_{\bar{z}\bar{z}z} &= -4\pi\epsilon e^{-2\rho/l_a}\left(i\frac{8Gj}{l_a} + 8Gm\right)\delta^{(2)}(z,\bar{z}), \\
{}^sR &= -8\pi\epsilon e^{-2\rho/l_a}\delta^{(2)}(z,\bar{z}) \\
&\quad + 32\pi\epsilon e^{-6\rho/l_a}\left((8Gm)^2 + \left(\frac{8Gj}{l_a}\right)^2\right)\delta^{(2)}(z,\bar{z}) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.26}$$

Then the total action is

$$
\begin{aligned}
I \simeq {}^{r}I_{\mathrm{EH}} - \frac{\epsilon}{4G} \int_{\rho_+}^{\rho_0} d\rho + \frac{\epsilon}{G}\left((8Gm)^2 + \left(\frac{8Gj}{l_a}\right)^2 \right) \int_{\rho_+}^{\rho_0} e^{-4\rho/l_a} d\rho + {}^{r}I_{\mathrm{CS}} \\
+ \frac{\alpha\epsilon j}{16 l_a^2} \int_\Sigma dx \sqrt{\gamma} \frac{(\frac{\partial F}{\partial \rho})^2}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2} \frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|} \\
= {}^{r}I_{\mathrm{EH}} + {}^{r}I_{\mathrm{CS}} \\
- \frac{c\epsilon}{6}\ln\frac{\beta}{a} + \frac{c\epsilon}{6}\ln\frac{\sqrt{r_+^2 - r_-^2}}{l_a} - \frac{c\epsilon}{6}\left((8Gm)^2 + \left(\frac{8Gj}{l_a}\right)^2 \right)\left(\frac{a^4}{\beta^4} - \frac{1}{4\pi^2 r_+^2 T^2} \right) \\
+ \frac{\alpha\epsilon j}{16 l_a^2} \int_\Sigma dx \sqrt{\gamma} \frac{(\frac{\partial F}{\partial \rho})^2}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2} \frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|} \\
\simeq {}^{r}I_{\mathrm{EH}} + {}^{r}I_{\mathrm{CS}} - \frac{c\epsilon}{6}\ln\frac{\beta}{a} + \frac{c\epsilon}{12}\ln(2\pi r_+ T) + \frac{c\epsilon}{6}\left((8Gm)^2 + \left(\frac{8Gj}{l_a}\right)^2 \right)\frac{1}{4\pi^2 r_+^2 T^2} \\
+ \frac{\alpha\epsilon j}{16 l_a^2} \int_\Sigma dx \sqrt{\gamma} \frac{(\frac{\partial F}{\partial \rho})^2}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2} \frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|}.
\end{aligned}
\tag{3.27}
$$

Here we have taken the UV cutoff $e^{\rho_0/l_a} \sim \beta/a$ as in [11] and β is the inverse of the temperature T. Thus we can obtain the HEE of the dual 2-dimensional CFT with gravitational Chern-Simons correction at finite temperature T

$$
\begin{aligned}
S_A \simeq \frac{c}{3}\ln\frac{\beta}{a} - \frac{c}{6}\ln(2\pi r_+ T) - \frac{c}{3}\left((8Gm)^2 + \left(\frac{8Gj}{l_a}\right)^2 \right)\frac{1}{4\pi^2 r_+^2 T^2} \\
- \frac{\alpha r_+ r_-}{64 G l_a^3} \int_\Sigma dx \frac{\frac{\partial F}{\partial \rho}\sqrt{(\frac{\partial F}{\partial x})^2 + g_{xx}(\frac{\partial F}{\partial \rho})^2}}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2} \frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|}.
\end{aligned}
\tag{3.28}
$$

Generally speaking, we need to solve the equation for the bulk static minimal surface to get the explicit contribution of the last term in eq. (3.28). For simplicity, here we would like to evaluate this term by making some approximations. Since the CFT system is located at the circle with radius $\rho = \rho_0 \gg l_a$, the bulk static minimal surface now is just a minimal curve which connect the boundary (two points) between subsystems A and B. The length of the minimal curve should have the same order of the arc length l_A of the subsystem A. When l_A is small compared to the length of the boundary, the bulk minimal curve is close to the arc of A on the boundary. Thus, the equation for the bulk minimal curve can be approximately evaluated by

$$
F = \rho(x) - \rho_0.
\tag{3.29}
$$

So $dF/d\rho = \partial F/\partial \rho = 1$ and $\partial F/\partial x = 0$. Consequently, we obtain the gravitational Chern-Simons term correction to HEE

$$
S_{\mathrm{CS}} \sim -\frac{\alpha\pi r_+ r_-}{64 G l_a^2}\frac{a}{\beta}.
\tag{3.30}
$$

It is obvious that S_{CS} is an finite correction to the HEE of subsystem A. Now let us consider two special cases. One is the extremal black hole case or zero temperature limit with $r_+ = r_-$, the other is the nonrotating BTZ black hole case with $r_- \to 0$.

When $r_+ = r_-$, $\beta \to \infty$, we have

$$S_A \simeq \frac{c}{3} \ln \frac{L}{a} - \frac{8c}{3} \frac{r_+^4}{l_a^4}. \tag{3.31}$$

Note that here we have restored the UV cutoff $e^{\rho_0/l_a} \sim L/a$ and $\rho \in [0, \rho_0]$ in the zero temperature limit. We can see that in this zero temperature limit, the gravitational Chern-Simons term correction to HEE vanishes and the universal part of HEE is the same as that in pure AdS$_3$ spacetime.

When $r_- \to 0$, we have

$$S_A \simeq \frac{c}{3} \ln \frac{\beta}{a} - \frac{c}{3} \ln \frac{r_+}{l_a} - \frac{c}{3}. \tag{3.32}$$

The HEE computed directly from the bulk minimal surface in nonrotating BTZ black hole metric is [11]

$$S_A \simeq \frac{c}{3} \ln \left(\frac{\beta}{\pi a} \sinh \frac{\pi l_A}{\beta} \right) = \frac{c}{3} \ln \left(\frac{\beta}{\pi a} \right) + \frac{c}{3} \ln \left(\sinh \frac{r_+ l_A}{2 l_a^2} \right). \tag{3.33}$$

Comparing eq. (3.32) with eq. (3.33), the leading terms give the same result, up to a small finite term $\sim \ln \pi$. In the low temperature phase, the second term in eq. (3.33) gives $\frac{c}{3} \ln \frac{r_+}{l_a}$, while the second term in eq. (3.32) gives a negative contribution $-\frac{c}{3} \ln \frac{r_+}{l_a}$, although both terms are small compared to the leading term. Also, like the zero temperature case, the contribution from the gravitational Chern-Simons term to HEE is zero up to the order of $h_{ij}^{(2)}$. From eq. (3.5), we can see that when $r_- \to 0$, the nonrotating BTZ black hole entropy is also not corrected due the bulk gravitational Chern-Simons term.

4 Conclusions and discussions

In this paper, we studied the gravitational Chern-Simons term correction to the HEE of the dual 2-dimensional CFT on the conformal boundary of AAdS$_3$ spacetime by using the off-shell Euclidean path integral approach. As we have expected, like the BTZ black hole entropy, although the bulk geometry does not change in the presence of the gravitational Chern-Simons term, the HEE is indeed modified by an finite term (up to the order of $h_{ij}^{(2)}$) at finite temperature case where the bulk is the BTZ black hole. This is because the presence of the gravitational Chern-Simons term violates the bulk diffeomorphism invariance of the gravitational action. In the AdS/CFT correspondence, the violation causes the gravitational anomaly (covariant anomaly) in the dual 2-dimensional CFT on the conformal boundary. Since the covariant anomaly modifies the central charges of CFT, it also should make a contribution to the HEE. While in the zero temperature and nonrotating BTZ black hole cases, the gravitational Chern-Simons term does not contribute to HEE in our calculation. However, the central charges of the dual CFT are corrected,

too. A possible explanation is that the large N correction caused by the bulk gravitational Chern-Simons term is not high enough to excite the NS-NS vacuum (since there is a mass gap between AdS$_3$ spacetime and the black hole spectrum) and the dual field of nonrotating BTZ black hole, so its entropy does not change. A support of this point is that from eq. (3.5), when $r_- = 0$, the nonrotating BTZ black hole entropy is not corrected. Anyway, this point needs a further study. It can be seen that, when the temperature T is zero or finite, our results in the case without gravitational Chern-Simons term correction reproduce those obtained from the geometric calculation. It should be stressed here that in the calculation, we did not actually solve the constraint equation $\delta \mathcal{A} = 0$. Instead, we just used the approximate UV-IR relations $\exp(\rho_0/l_a) = L/a$ at zero temperature and $\exp(\rho_0/l_a) = \beta/a$ at finite temperature, these relations are enough to show the gravitational Chern-Simons term correction to HEE in the leading order. To find the full exact result, one need to solve $\delta \mathcal{A} = 0$ and find out the exact UV-IR relation.

We want to point out that generally, the HEE is not the same as the black hole entropy when there is a black hole in the bulk. In fact, as has been shown in [17], the bulk minimal surfaces are generated either by null curves (with black hole in bulk) or spacelike curves (without black hole in bulk), and the area of the bulk minimal surface acts as the entropy bound (such as the covariant entropy bound [51]) of the CFT system. Thus the HEE indicates the maximum entropy of the CFT system.

Acknowledgments

A Explicit derivation of the Chern-Simons term correction

For AdS$_3$ spacetime and BTZ black hole, eq. (3.6) can be expressed as

$$ds^2 = d\rho^2 + g_{tt}dt^2 + 2g_{tx}dtdx + g_{xx}dx^2, \tag{A.1}$$

where $g_{tt} = h_{tt}$, $g_{tx} = h_{tx}$, $g_{xx} = h_{xx}$ and $g = h \simeq -\frac{1}{4}e^{4\rho/l_a}$. Set the bulk static minimal surface (denoted by Σ) be

$$x = x(\rho) \quad \text{or} \quad F[x(\rho), \rho] = 0. \tag{A.2}$$

One of its normal vectors is

$$n^1 = \left(0, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial \rho}\right). \tag{A.3}$$

There is another normal vector of the bulk static minimal surface

$$n^2 = (f, 0, 0), \tag{A.4}$$

where f is some function which can be determined by the normalization condition. From eq. (A.3) and eq. (A.4) we can read the nonvanishing components of these normal vectors

$$n_x^1 = \frac{\partial F}{\partial x}, \quad n_\rho^1 = \frac{\partial F}{\partial \rho}, \quad n_t^2 = f. \tag{A.5}$$

Thus we have

$$
\begin{aligned}
n^1 n_1 &= n_t^1 n_1^t + n_x^1 n_1^x + n_\rho^1 n_1^\rho = n_x^1 (g^{xx} n_{1x} + g^{xt} n_{1t}) + g^{\rho\rho} n_1^\rho n^{1\rho} \\
&= g^{xx} \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial \rho} \right)^2.
\end{aligned} \tag{A.6}
$$

and

$$n^2 n_2 = n_t^2 n_2^t = g^{tt} f^2. \tag{A.7}$$

Then we get the unit normal vectors

$$
\begin{aligned}
\hat{n}^1 &= \frac{1}{\sqrt{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2}} \left(0, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial \rho} \right), && \text{and} \quad \hat{n}^2 = \frac{1}{\sqrt{-g^{tt}}} (1, 0, 0), \\
\hat{n}_1 &= \frac{1}{\sqrt{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2}} \left(g^{tx} \frac{\partial F}{\partial x}, g^{xx} \frac{\partial F}{\partial x}, \frac{\partial F}{\partial \rho} \right), && \text{and} \quad \hat{n}_2 = \frac{1}{\sqrt{-g^{tt}}} (g^{tt}, g^{xt}, 0),
\end{aligned} \tag{A.8}
$$

which satisfy $\hat{n}^1 \hat{n}_1 = 1$ and $\hat{n}^2 \hat{n}_2 = -1$.

The singular part of the gravitational Chern-Simons term is [26]

$$^s I_{\text{CS}} = -\frac{\alpha\epsilon}{32G} \int_{\mathcal{M}} d^3 x \sqrt{-g} \, \Gamma_{\alpha\beta\mu} \epsilon^{\mu\nu\lambda} [(n^\beta n_\nu)(n^\alpha n_\lambda) - (n^\beta n_\lambda)(n^\alpha n_\nu)] \delta_\Sigma, \tag{A.9}$$

where

$$\delta_\Sigma = \frac{\delta(t - C)\delta(F[x(\rho), \rho])}{\text{measure of the } t, \rho \text{ components}} \tag{A.10}$$

is the 2-d delta function which is nonvanishing only on the bulk static minimal surface Σ, C is some const. time and

$$\delta(F[x(\rho), \rho]) = \frac{\delta(\rho - \bar{\rho})}{|\frac{dF}{d\rho}|}, \tag{A.11}$$

where $\bar{\rho}$ is the value of ρ on Σ. Integrating out the delta function in eq. (A.9), we obtain

$$^s I_{\text{CS}} = -\frac{\alpha\epsilon}{32G} \int_{\Sigma} dx \sqrt{\gamma} \, \Gamma_{\alpha\beta\mu} \epsilon^{\mu\nu\lambda} [(n^\beta n_\nu)(n^\alpha n_\lambda) - (n^\beta n_\lambda)(n^\alpha n_\nu)] \frac{1}{|\frac{dF}{d\rho}|}, \tag{A.12}$$

where $\sqrt{\gamma} dx = \sqrt{(d\rho/dx)^2 + g_{xx}} dx$ is the invariant measure on Σ.

Making coordinate transformation in the Gaussian normal coordinate eq. (A.1), labeling $x^0 = t = -i\tau$, $j \to ij$ and $x^1 = x$. Defining

$$w = \frac{1}{2}(\tau + ix) \quad \text{and} \quad \bar{w} = \frac{1}{2}(\tau - ix), \tag{A.13}$$

then we have $\tau = w + \bar{w}$ and $x = -i(w - \bar{w})$. Thus we can rewrite eq. (A.1) as

$$ds^2 = d\rho^2 - (h_{00} + 2h_{01} + h_{11})dw^2 - 2(h_{00} - h_{11})dwd\bar{w} - (h_{00} - 2h_{01} + h_{11})d\bar{w}^2. \quad (A.14)$$

Like before, we further introduce a singular coordinate transformation $w = z^n = z^{1+\epsilon}$ and expand it in the linear order of ϵ. After that we obtain

$$
\begin{aligned}
ds^2 &= d\rho^2 - (h_{00} + 2h_{01} + h_{11})h_{zz}dz^2 - 4(h_{00} - h_{11})h_{z\bar{z}}dzd\bar{z} - (h_{00} - 2h_{01} + h_{11})h_{\bar{z}\bar{z}}d\bar{z}^2 \\
&\equiv d\rho^2 + g_{zz}(\rho, z)dz^2 + 2g_{z\bar{z}}(\rho, z, \bar{z})dzd\bar{z} + g_{\bar{z}\bar{z}}(\rho, \bar{z})d\bar{z}^2. \quad (A.15)
\end{aligned}
$$

Then the unit normal vectors $\hat{n}_1$ and $\hat{n}_2$ become

$$
\begin{aligned}
\hat{n}_1 &= \frac{1}{\sqrt{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2}}\left[\frac{i}{2}\left(g^{tx}\frac{\partial F}{\partial x} + g^{xx}\frac{\partial F}{\partial x}\right)\partial_z + \frac{i}{2}\left(g^{tx}\frac{\partial F}{\partial x} - g^{xx}\frac{\partial F}{\partial x}\right)\partial_{\bar{z}} + \frac{\partial F}{\partial \rho}\partial_\rho\right], \\
\hat{n}_2 &= \frac{1}{\sqrt{-g^{tt}}}\left[\frac{i}{2}(-g^{tt} + g^{xt})\partial_z - \frac{i}{2}(g^{tt} + g^{xt}\partial_{\bar{z}})\right], \quad (A.16)
\end{aligned}
$$

The nonvanishing components of $\Gamma_{\alpha\beta\lambda}$ are

$$
\begin{aligned}
\Gamma_{z\bar{z}z} &= -\frac{1}{2}g_{\bar{z}z,z}, & \Gamma_{z\bar{z}\bar{z}} &= \frac{1}{2}g_{\bar{z}z,\bar{z}}, & \Gamma_{z\rho z} &= \frac{1}{2}g_{zz,\rho}, \\
\Gamma_{z\rho\bar{z}} &= \frac{1}{2}g_{z\bar{z},\rho}, & \Gamma_{\bar{z}\rho z} &= \frac{1}{2}g_{z\bar{z},\rho}, & \Gamma_{\bar{z}\rho\bar{z}} &= \frac{1}{2}g_{\bar{z}\bar{z},\rho}. \quad (A.17)
\end{aligned}
$$

For AdS$_3$ spacetime and BTZ black hole, up to the order of $h^{(2)}$, $g_{zz,\rho} = 0$ and $g_{\bar{z}\bar{z},\rho} = 0$. Meanwhile, both $g_{\bar{z}z,z}$ and $g_{\bar{z}z,\bar{z}}$ contain an ϵ factor, thus they do not contribute to $^sI_{\mathrm{CS}}$ up to the order of ϵ. Substituting eq. (A.16) and eq. (A.17) into eq. (A.12), we obtain

$$^sI_{\mathrm{CS}} = \frac{\alpha\epsilon}{32G}\int_\Sigma dx\sqrt{\gamma}\frac{\epsilon^{z\bar{z}\rho}}{2}(h_{tt} - h_{xx})_{,\rho}(\hat{n}_1^\rho)^2\left[\frac{h_{tx}}{h}(h_{tt} + h_{xx}) + \frac{h_{tx}}{h}\left(h_{xx} + \frac{h_{tx}^2}{h_{xx}}\right)\right]\frac{1}{|\frac{dF}{d\rho}|}. \quad (A.18)$$

For the AdS$_3$ spacetime, eq. (A.18) vanishes since h_{tx} is zero, i.e., the gravitational Chern-Simons term does not alter the HEE in the zero temperature case. While for the BTZ black hole, up to the order of $h^{(2)}$, we have

$$^sI_{\mathrm{CS}} = \frac{\alpha\epsilon j}{16l_a^2}\int_\Sigma dx\sqrt{\gamma}\frac{(\frac{\partial F}{\partial \rho})^2}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2}\frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|}. \quad (A.19)$$

With the help of eq. (2.20), we obtain the gravitational Chern-Simons term correction to HEE at the finite temperature case

$$S_{\mathrm{CS}} = -\frac{\alpha r_+ r_-}{64Gl_a^3}\int_\Sigma dx\frac{\frac{\partial F}{\partial \rho}\sqrt{(\frac{\partial F}{\partial x})^2 + g_{xx}(\frac{\partial F}{\partial \rho})^2}}{g^{xx}(\frac{\partial F}{\partial x})^2 + (\frac{\partial F}{\partial \rho})^2}\frac{e^{-2\rho/l_a}}{|\frac{dF}{d\rho}|}. \quad (A.20)$$

In the above calculation, we have used the following result:
Under general coordinate transformation $x \to x'$,

$$\sqrt{-g}\epsilon^{\mu\nu\lambda}d^3x \rightarrow \sqrt{-g'}\epsilon'^{\mu\nu\lambda}d^3x'$$
$$= \left|\frac{\partial x}{\partial x'}\right|\sqrt{-g}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial x'^\lambda}{\partial x^\gamma}\epsilon^{\alpha\beta\gamma}\left|\frac{\partial x'}{\partial x}\right|d^3x, \qquad (A.21)$$

where $g' = 4g$ in this case, which gives

$$\epsilon^{\rho z \bar{z}} = \frac{i}{2\sqrt{-g}}, \qquad (A.22)$$

and

$$\sqrt{-g'}\epsilon^{\rho z \bar{z}}d\rho dz d\bar{z} = 2\sqrt{-g}\frac{i}{2\sqrt{-g}}d\rho dz d\bar{z} = id\rho dz d\bar{z}. \qquad (A.23)$$

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